Number Systems

By: Sagar Aggarwal

Contents

Introduction

- A number 'r' is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
- Every rational number can have infinite number of equivalent rational numbers. For example,

  $\frac{1}{2} = \frac{2}{4} = \frac{5}{10} = \frac{25}{50}$ and so on are equivalent rational numbers, However, we can say that $\frac{p}{q}$ is a rational number, or when we represent $\frac{p}{q}$ on the number line, we assume that $q \neq 0$ and that p and q have no common factors other than 1 (that is, p and q are co-prime). So, on the number line, among the infinitely many fractions equivalent to $\frac{1}{2}$ to represent all of them.

Irrational Numbers

- A number 's' is called irrational, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and q≠0.
- Some examples of irrational numbers are: $\sqrt{2}, \sqrt{3}, \sqrt{15}$, π, 0.10110111011110111...
- Let us now locate $\sqrt{2}$ on the number line:

Consider a square OABC, with each side 1 unit in length (see Fig. 1.1)

Then you can see that by the Pythagoras Theorem, OB= $\sqrt{1^2+1^2} = \sqrt{2}$

This is how we represent $\sqrt{2}$ on the number line by point P (see Fig. 1.2):

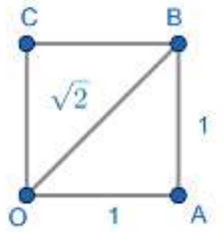


Fig. 1.1

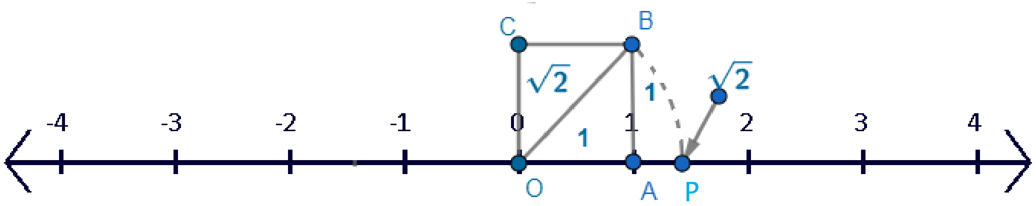


Fig. 1.2

Real Number and their Decimal Expansions

- Of all the rationals of the form $\frac{p}{q}$, on division of p by q, if:

  Case (i): The remainder becomes zero:
  The decimal expansion of such numbers is terminating. E.g. $\frac{1}{2} = 0.5$

  Case (ii): The remainder never becomes zero:
  The decimal expansion of such numbers is non-terminating and recurring (repeating).

  E.g. $\frac{1}{3} = 0.\bar{3}$

Real Number and their Decimal Expansions (Contd..)

- Let us consider some examples of non-terminating and repeating decimal expansions.
- **Example 1**: Show that $1.272727... = 1.\overline{27}$ can be expressed in the form $\frac{p}{q}$, where p and q are integers and q≠0.

  Solution: Let x=1.272727... Since two digits are repeating, we multiply x by 100 to get

  100x=127.2727...

  ⇒100x=126+1.272727..

  ⇒100x−x=126

  ⇒99x=126

  ⇒x= $\frac{126}{99} = \frac{14}{11}$

Real Number and their Decimal Expansions (Contd..)

- Note that $\sqrt{2} = 1.41421356237309504880168872420 96...$

  π=3.14159265358979323846264338327950...

  These are non-terminating and non-recurring decimal expansions and they are irrational numbers.

- So, from above discussions, we can state that:
  - The decimal expansion of a rational number is either terminating or non terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.
  - The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.

Representing Real Numbers on the Number Line

- Any real number has a decimal expansion. This helps us to represent it on the number line.
- For example, we want to locate 2.665 on the number line.

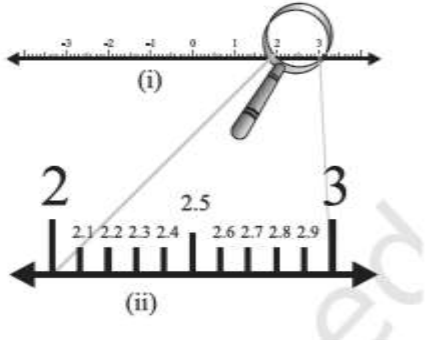
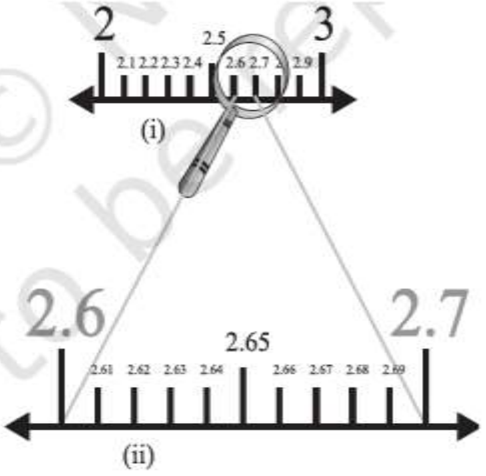
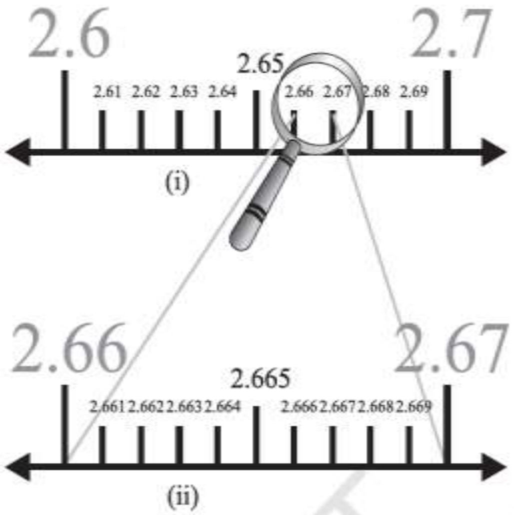


Fig. 1.3

Between 2 and 3



Between 2.6 and 2.7



Between 2.66 and 2.67

Representing Real Numbers on the Number Line (Contd..)

- We know that 2.665 lies between 2 and 3 (See Fig. 1.3 on previous page). So we closely look at the portion of number line between 2 and 3. Suppose we divide this into 10 equal parts and mark each point of division as "2.1, 2.2, 2.3 till 2.9" as shown in **"Between 2 and 3" part of Fig. 1.3.**
- We further know that 2.665 will lie between 2.6 and 2.7. So we now divide this part of number line into 10 equal parts and mark each point of division as "2.61, 2.62, 2.63 till 2.69" as shown in **"Between 2.6 and 2.7" part of Fig. 1.3.**
- We also know that 2.665 will lie between 2.66 and 2.67. So we now divide this part of number line into 10 equal parts and mark each point of division as "2.661, 2.662, 2.663 till 2.669" as shown in **"Between 2.66 and 2.67" part of Fig.1.3.**
- This process of visualisation of representing of numbers on the number line, through a magnifying glass, is called as **the process of successive magnification.**

Operations on Real Numbers

- We know that rational numbers satisfy the commutative, associative and distributive laws for addition and multiplication. Moreover, if we add, subtract, multiply or divide (except by zero) two rational numbers, we still get a rational number.
- Also note that, irrational numbers also satisfy the commutative, associative and distributive laws for addition and multiplication. However, the sum, difference, quotients and products of irrational numbers are not always irrational. For example, $(\sqrt{6})+(-\sqrt{6}),(\sqrt{2})-(\sqrt{2}) and \frac{\sqrt{17}}{\sqrt{17}}$ are rationals.
- Also, the below facts are true about operations on real numbers:
  i. The sum or difference of a rational number and an irrational number is irrational.
  ii. The product or quotient of a non-zero rational number with an irrational number is rational
  iii. If we add, subtract, multiply or divide two irrationals, the result may rational or irrational.

Operations on Real Numbers (Contd..)

- Let us represent $\sqrt{3.5}$ geometrically.

Mark the distance 3.5 units from a fixed point A on a given line to obtain a point B such that AB=3.5 units (see Fig. 1.4). From B, mark a distance of 1 unit and mark the new point as C. Find the mid-point of AC and mark the point as O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then

BD= $\sqrt{3.5}$.

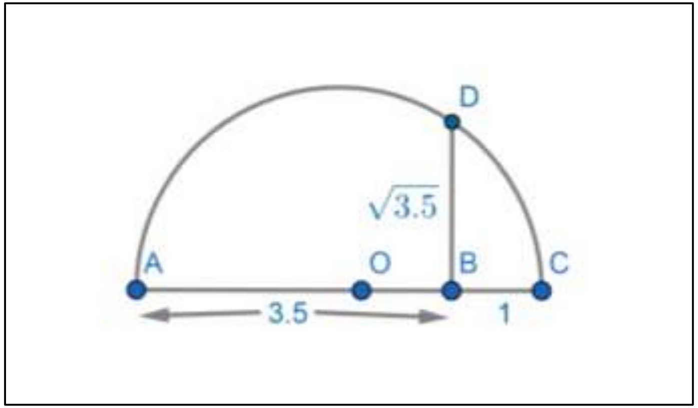


Fig. 1.4

Operations on Real Numbers (Contd..)

- More generally, let us find $\sqrt{x}$, for any positive real number x, represent it geometrically.

Mark the distance x units from a fixed point A on a given line to obtain a point B such that AB=x units (see Fig. 1.5). From B, mark a distance of 1 unit and mark the new point as C. Find the mid-point of AC and mark the point as O. Draw a semicircle with centre O and radius OC. Draw a line perpendicular to AC passing through B and intersecting the semicircle at D. Then

BD= $\sqrt{x}$. We can prove this as follows:

△OBD is a right angled triangle where:

OD=OC=OA= $\frac{x+1}{2}$ units. Now, OB=x− $\frac{x+1}{2} = \frac{x-1}{2}$ units

So, by Pythagoras Theorem,

$BD^2=OD^2-OB^2=\left(\frac{x+1}{2}\right)^2-\left(\frac{x-1}{2}\right)^2=\frac{4x}{4}=x$. So BD= $\sqrt{x}$

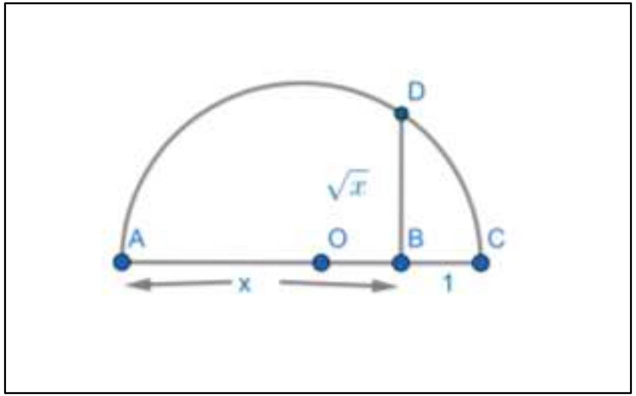


Fig. 1.5

Operations on Real Numbers (Contd..)

- Let a and b be positive real numbers. Then:

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$

(iv) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{c} + \sqrt{d}\right) = \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd}$

(vi) $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

Laws of Exponents for Real Numbers

- Let a, b>0 be real numbers and p and q be rational numbers. Then, we have:

(i) $a^p . a^q = a^{p+q}$

(ii) $(a^p)^q = a^{pq}$

(iii) $\frac{a^p}{a^q} = a^{p-q}$

(iv) $a^p b^p = (ab)^p$

Also note that, let a>0 be a real number and m, n are integers such that m and have no common factors other than 1 and n>0, then:

$$a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$$

Summary

- A number r is called a rational number, if it can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
- A number s is called a irrational number, if it cannot be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
- The decimal expansion of a rational number is either terminating or non-terminating recurring. Moreover, a number whose decimal expansion is terminating or non-terminating recurring is rational.
- The decimal expansion of an irrational number is non-terminating non-recurring. Moreover, a number whose decimal expansion is non-terminating non-recurring is irrational.
- All the rational and irrational numbers make up the collection of real numbers.
- There is a unique real number corresponding to each real number, there is a unique point on the number line.
- If r is rational and s is irrational, then r+s and r–s are irrational numbers, and rs and $\frac{r}{s}$ are irrational numbers, $r \neq 0$.

Summary (Contd..)

- For positive real numbers a and b, the following identities hold:

(i) $\sqrt{ab} = \sqrt{a}\sqrt{b}$

(ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

(iii) $\left(\sqrt{a} + \sqrt{b}\right)\left(\sqrt{a} - \sqrt{b}\right) = a - b$

(iv) $\left(a + \sqrt{b}\right)\left(a - \sqrt{b}\right) = a^2 - b$

(v) $\left(\sqrt{a} + \sqrt{b}\right)^2 = a + 2\sqrt{ab} + b$

- Let a, b>0 be a real number and p and q be rational numbers. Then:

(i) $a^p . a^q = a^{p+q}$

(ii) $\left(a^p\right)^q = a^{pq}$

(iii) $\frac{a^p}{a^q} = a^{p-q}$

(iv) $a^p b^p = (ab)^p$

THANK YOU